ON THE CONSTRUCTION OF SOME CLASSES OF NEIGHBOUR DESIGNS

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INTRODUCTION

The concept of 'Neighbour' designs was introduced by Rees [4]. Such designs have use mainly in the field of Serology, and some of them can be used for animal husbandry experiments. A neighbour design is defined as follows :—

There are v symbols (antigens) arranged in b circular plates (blocks) such that :

(*i*) each block has k symbols, not necessarily all distinct;

(*ii*) each symbol appears r times in the design;

(*iii*) every symbol is a 'neighbour' of every other symbol precisely λ times.

A neighbour design with $k<v$ may be called an incomplete neighbour design. Methods of construction of incomplete neighbour designs of some types have been discussed by Rees [4] and more recently, by Hwang [2]. It was shown by Lawless [3] that some series of Balanced Incomplete Block (*BIB*) designs are also neighbour designs.

The purpose of the present paper is to discuss several methods of construction of incomplete neighbour designs, using the method of differences, and also through *BIB* designs.

2. SOME PRELIMINARIES

In this section we set out the basic principles on which the construction of neighbour designs is based. The principle is originally

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due to Rees [4], and consists in choosing some 'basic' blocks satisfying a certain property. The complete design is then obtained by 'developing' each of the basic blocks, as is done for obtaining incomplete block designs.

Consider a block containing k elements

$$(i_1, i_2, \ldots, i_k). \qquad \ldots(2.1)$$

The 'forward' differences arising from this block are

$$F_k = (i_2 - i_1,\ i_3 - i_2,\ \ldots,\ i_k - i_{k-1},\ i_1 - i_k). \qquad \ldots(2.2)$$

Similarly, the 'backward' differences arising from the block (2.1) are

$$B_k = (i_1 - i_2,\ i_2 - i_3,\ \ldots,\ i_{k-1} - i_k,\ i_k - i_1). \qquad \ldots(2.3)$$

Clearly, $B_k = -F_k$.

With the above definitions, Rees [4] laid down the following principle for the construction of neighbour designs.

Consider a module of v elements, numbered say, 0, 1,...$v-1$ and lel there exist a set of t 'basic' blocks $(i_{1j}, i_{2j}, \ldots, i_{kj})$, $j=1, \ldots, t$, each block containing k (not necessarily distinct) elements of the module. These t basic blocks, when developed modulo v generate a neighbour design if the following conditions are satisfied :

(*i*) Among the totality of forward and backward differences reduced mod v, arising from the t basic blocks, every non-zero element of the module occurs equally frequently.

(*ii*) The sum of the forward differences arising from each basic block is zero.

Condition (*ii*) is obviously satisfied by any block and thus, it is enough to satisfy condition (*i*) in order to construct a neighbour design.

A slight generalisation of this principle can be made by introducing the notion of 'pure forward' (or backward) differences and 'mixed forward' (or backward) differences. (Bose, [1]).

Consider a module containing m elements. To each element, let there correspond n elements and let the symbols corresponding to the u-th element be $u_1, u_2, \ldots, u_n$. Thus, we have exactly $m\,n$ symbols. Symbols with the same suffix j will be said to belong to the j-th class.

Let a block S contain $k = \sum_{i=1}^{n} p_i$. Let the symbols belonging to the i-th class in S be $a_i(^1)$, $a_i(^2)$, …, $a_i(^{p_i})$ and those of the j-th class in S be $b_j(^1)$, …, $b_j(^{p_j})$.

A forward difference from S will be called a 'pure forward difference' of the type (i, i) if it is the difference between the symbols of the i-th class, *i.e.* of the type $a_i(^u) - a_i(^v)$, $u \neq v$. Similarly, a backward difference from S will be called a pure backward difference if the difference is taken between the symbols of the same class.

Again, a forward (backward) difference from S will be called a 'mixed' forward (backward) difference of the type (i, j) if the difference is of the type $a_i(^u) - b_j(^v)$. Clearly, the mixed forward difference of the type (i, j) will be the negative of the mixed backward difference of the type (j, i).

Now, let us consider a set of t basic blocks each containing k distinct symbols, satisfying the following conditions :

(*i*) Among the totality of pure forward and pure backward differences of the type (i, i), arising from all the t basic blocks, each non-zero element of the module is repeated equally frequently (λ times each), independently of i.

(*ii*) Among the totality of mixed forward and mixed backward differences of the type (i, j) arising from the t basic blocks, every element of the module appears λ times, independently of i and j.

(*iii*) Among the $k\,t$ symbols occurring in the t basic blocks, exactly r symbols belong to each of the n classes.

Then by developing these basic blocks we get a neighbour design with parameters :

$$v = m\,n,$$
$$b = m\,t,$$
$$r = k\,t/n,\ k,\ \lambda.$$

3. NEIGHBOUR DESIGNS IN RELATION TO BIB DESIGNS

Let there exist a neighbour design with parameters v, b, r, k, λ. Then, it is easy to see that these parameters are related by the following relations :

$$v\,r = b\,k, \qquad \lambda(v-1) = 2\,r. \qquad \ldots(3.1)$$

Now, if the circular blocks of the neighbour design are to form a BIB design with parameters v^*, b^*, r^*, k^*, λ^*, then we must have :

$$v^* = v,$$
$$b^* = \lambda\,v\,(v-1)/2k = b,$$
$$r^* = \lambda\,(v-1)/2,$$
$$k^* = k,$$
$$\lambda^* = \lambda\,(k-1)/2. \qquad \ldots(3.2)$$

However, since the relations (3.1) are merely necessary, any *BIB* design with parameters (3.2) may or may not be a neighbour design. It is therefore of some interest to characterise *BIB* designs which are neighbour designs too. It is easily seen that for $\lambda=1$ we have the BIB design :

$$v^*=2\,m\,k^*+1,$$
$$b^*=m\,(2\,m\,k^*+1),$$
$$r^*=m\,k^*,$$
$$\lambda^*=(k^*-1)/2.$$

A solution for this *BIB* design exists whenever v^* is a prime power (Sprott, [6]) and such a solution is a neighbour design too, as shown by Lawless [3]. Similarly, with $\lambda=2$ in (3.2) we have the *BIB* design

$$v^*=m\,k^*+1,$$
$$b^*=m\,(m\,k^*+1),$$
$$r^*=m\,k^*,\ k^*,$$
$$\lambda^*=k^*-1$$

and a solution of this design exists whenever $v^*=m\,k^*+1$ is a prime power. Such a solution is also a neighbour design (Lawless [3]). There are probably many more *BIB* designs which are neighbour designs also. To this end, we have the following

Theorem 3.1. Let there exist a *BIB* design with parameters v^*, b^*, r^*, $k^*=3$, λ^* for which a difference set solution exists. Then, such a solution is also a solution for a neighbour design with parameters $v=v^*$, $b=b^*$, $r=r^*$, $k=k^*$, $\lambda=\lambda^*$. Conversely, a solution for a neighbour design with $k=3$ is also a solution for a *BIB* design.

Proof. The result follows almost immediately because the totality of forward and backward differences arising from all the basic blocks are also the *all possible* differences arising from these blocks, since $k^*=3$.

Combining the above result with the well known results on *BIB* designs with $k^*=3$, $\lambda^*=1$ (Bose, [1]) we have the following corollaries :

Corollary 3.1. Let $v=6\,t+1$ be a prime power. Then there exists a neighbour design with parameters $v=6\,t+1$, $b=t\,(6\,t+1)$, $r=3\,t$, $k=3$, $\lambda=1$.

Corollary 3.2. Let $v=6\,t+3$. Then there exists a neighbour design with parameters

$$v=6\,t+3,\ b=(3\,t+1)\,(2\,t+1),$$
$$r=3\,t+1,\ k=3,\ \lambda=1.$$

4. Another Class of Neighbour Designs with $k=3$

In this section we give a method of construction of another series of neighbour designs with $k=3$. Let v be a prime power of the form $4t+3$ and x be a primitive element of $GF(v)$. Then, it is easy to see that x^o+x is either an even power of x (say x^{2u}) or an odd power of x (say, x^{2m+1}). In each of these cases, a neighbour design with $k=3$ can be constructed. The result is contained in the following

Theorem 4.1. (*i*) If $x^o+x=x^{2u}$, the basic blocks $(0, x^i, x^{2u+i})$, $i=0, 2, 4, \ldots, 4t$, when developed give a neighbour design with parameters

$$v=4t+3,$$
$$b=(2t+1)(4t+3),$$
$$r=3(2t+1),$$
$$k=3, \lambda=3. \qquad \ldots(4.1)$$

(*ii*) If $x^o+x=x^{2m+1}$, the basic blocks (o, x^{i+1}, x^{2m+1+i}), $i=o, 2, \ldots, 4t$ when developed give a neighbour design with parameters given in (4.1).

Proof. We prove only (*i*) —the proof for (*ii*) follows more or less on the same lines.

Consider the set of basic blocks

$$(o, x^i, x^{2u+i}), \; i=0, 2, 4, \ldots, 4t$$

where
$$x^{2u}=x^o+x.$$

The totality of forward and backward differences arising out of these $(2t+1)$ basic blocks can be written as:

(*i*) $\pm x^o, \pm x^2, \pm x^4, \ldots, \pm x^{4t}$;

(*ii*) $\pm(x^{2u}-x^o), \pm(x^{2u+2}-x^2), \pm(x^{2u+4}-x^4), \ldots, \pm(x^{2u+4t}-x^{4t})$;

(*iii*) $\pm x^{2u}, \pm x^{2u+2}, \ldots, \pm x^{2u+4t}$.

Clearly, among the differences of types (*i*) and (*iii*) taken together, each non-zero element of $GF(v)$ occurs exactly twice. Letting $z=x^{2u}-x^o$, the type (*ii*) differences can be written as

$$\pm x^0 z, \pm x^2 z, \ldots, \pm x^{4t} z.$$

Thus, among the type (*ii*) differences, every non-zero element of $GF(v)$ occurs exactly once, givihg the value of $\lambda=3$. The expressions for other parameters are obvious.

A Remark. By theorem 3.1, the basic blocks given in theorem 4.1 give a *BIB* design with

$$v^*=4\,t+3,\ b^*=(2t+1)(4t+3),\ r^*=3(2t+1),\ k^*=3=\lambda^*.$$

Also, as pointed out by Saha [5], the designs given by theorem 4.1 can be used as change-over designs balanced for first residual effects if the blocks of the design are written as columns and the columns of the arrangement are treated as sequences and rows as periods.

5. Some Other Classes of Neighbour Designs

In this section we construct some more classes of neighbour designs. In the designs discussed so far, each symbol was allowed to appear in a block at most once. However, this condition is not necessary for a neighbour design and, in fact many designs can be constructed in which a symbol appears more than once in a block. We consider such designs in this section.

5.1. Let $v(=4t+3)$ be a prime power and x, a primitive element of $GF(v)$. Consider the set of basic blocks

$$(o,\ x^{2u},\ o,\ x^{2u+2}),\quad u=o,\ 1,\ldots\ldots,\ 2t. \qquad \ldots(5.1)$$

Then we can easily prove

Theorem 5.1. The $(2t+1)$ basic blocks in (5.1) when developed give a neighbour design with $v=4t+3$, $b=(2t+1)(4t+3)$, $r=4(2t+1)$, $k=4$, $\lambda=4$.

Let, as before, $v(=4t+3)$ be a prime power and x, a primitive element of $GF(v)$. Consider the following set of initial blocks :

$$\begin{array}{l}(x^o,\ x^2,\ x^o,\ x^4,\ x^6,\ x^8,\ \ldots\ldots,\ x^{4t})\\(x^o,\ x^2,\ x^4,\ x^2,\ x^6,\ x^8,\ \ldots\ldots,\ x^{4t})\\(x^o,\ x^2,\ x^4,\ x^6,\ x^4,\ x^8,\ \ldots\ldots,\ x^{4t})\\\vdots\\(x^o,\ x^2,\ x^4,\ x^6,\ \ldots\ldots,\ x^{4t-2},\ x^{4t},\ x^{4t-2})\\(x^{4t},\ x^2,\ x^4,\ x^6,\ldots\ldots,\ x^{4t},\ x^o).\end{array} \qquad \ldots(5.2)$$

Thus we have the following

Theorem 5.2. The $(2t+1)$ basic blocks given in (5.2) when developed give a neighbour design with

$$v=4\,t+3,\quad b=(2t+1)\,(4t+3),\quad r=(2t+1)\,(2t+2),$$
$$k=2\,t+2,\quad \lambda=2\,t+2.$$

Proof. We show only the value of λ. The forward and backward differences arising from the basic blocks (5.2) are of the following two types:

(*i*) $\pm(x^{2u+2}-x^{2u})$ $u=0, 1, \ldots\ldots, 2t$

(*ii*) $\pm(x^{2u+4}-x^{2u})$ $u=0, 1, \ldots\ldots, 2t.$

The differences (*i*) each appear $(2t+1)$ times while differences (*ii*) each appears once. Also, among the differences of type (*i*), every non-zero element of $GF(v)$ appears exactly once and so also among the differences of type (*ii*).

Thus, $\lambda=2t+1+1=2t+2.$

5.3. Two more series of neighbour designs can be constructed for $k=v-1$. The following two theorems relate to these designs and can be proved easily :

Theorem 5.3. Let $v(=4t+3)$ be a prime power and x, a primitive element of $GF(v)$. The basic block

$$(0, x^0, 0, x^2, 0, x^4, \ldots\ldots 0, x^{4t})$$

when developed gives a neighbour design with

$$v=4t+3=b, \quad r=4t+2=k, \quad \lambda=2.$$

Theorem 5.4. Let $v(=4t+1)$ be a prime power and x, a primitive element of $GF(v)$. The basic blocks $(0, x^o, 0\ x^2, \ldots\ldots, 0, x^{4t+2})$ and $(0, x, 0, x^3, \ldots\ldots, 0, x^{4t-1})$ when developed give neighbour design with $v=4t+1$, $b=8t+2$, $r=8t$, $k=4t$, $\lambda=4$.

6. Trial and Error Solutions

Some other neighbour designs have been constructed through trial and error. The parameters of these designs and basic blocks are given below :

Sl. No.	v	b	r	k	λ	*Basic Blocks*
1.	7	21	12	4	4	(1, 3. 2, 6), (2, 6, 4, 5) (4, 5, 1, 3) mod 7
2.	7	21	15	5	5	(0, 1, 4, 6, 5), (0, 2, 1, 5, 3), (0, 4, 2, 3, 6) mod 7
3.	13	39	12	4	2	(1, 2, 4, 8), (3, 6, 12, 11), (4, 8, 3, 6) mod 13
4.	37	74	18	9	4	(1, 16, 34, 26, 9, 33, 10, 12, 7) (6, 22, 19, 8, 17, 13, 23, 35, 5) mod 37.

SUMMARY

The purpose of the present paper is to provide methods of construction of some series of Neighbour designs.

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